Lecture 3

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1 Overview

In this lecture, we introduce the concept of nonexpansiveness and several variants. Nonexpansive operators are Lipschitz continuous operators with Lipschitz constant 1. They play an important rols in convex optimization since many problems reduce to finding fixed points of nonexpansive operators.

Suppose the solution set to an optimization problem is C (usually there are more than one solution). An algorithm typilcally goes with

$$x_{n+1} = Tx_n, \quad n = 1, 2, \cdots$$

where $T : \mathcal{H} \to \mathcal{H}$ is an operator. Ideally, $x_n \to x \in C$, i.e. $\|x_n - x\|_2 \to 0$.

Then the solution set is the fixed point set of T, $\mathrm{Fix}T = \{x \in \mathcal{H} \mid x = Tx\}$.

2 Nonexpansive and firmly nonexpansive operators

An operator $T : \mathcal{H} \to \mathcal{H}$ is

- **nonexpansive** if $\forall (x, y) \in \mathcal{H}^2$, $\|Tx - Ty\| \leq \|x - y\|$. That is, T is 1-Lipschitz continuous.
  - T is nonexpansive $\Rightarrow$ T is continuous.

  Proof: Suppose $x_n \to x$, then $\|Tx_n - Tx\| \leq \|x_n - x\| \to 0$. Then $Tx_n \to Tx$.
- **quasi nonexpansive** if $\forall x \in \mathcal{H}$, $\forall y \in \mathrm{Fix}T$, $\|Tx - y\| \leq \|x - y\|$
- **firmly nonexpansive** if $\forall (x, y) \in \mathcal{H}^2$, $\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\| \leq \|x - y\|^2$, where Id denotes the identity operator.
  - Or equivalently, $\forall (x, y) \in \mathcal{H}^2$, $\langle x - y \mid Tx - Ty \rangle \geq \|Tx - Ty\|2$.
- **firmly quasi nonexpansive** if $\forall x \in \mathcal{H}$, $\forall y \in \mathrm{Fix}T$, $\|Tx - y\|^2 + \|Tx - x\|^2 \leq \|x - y\|^2$

- Or equivalently, $\forall x \in \mathcal{H}$, $\forall y \in \mathrm{Fix}T$, $\langle x - y \mid Tx - y \rangle \geq \|Tx - y\|^2$.
- That is, $\forall x \in \mathcal{H}$, $\forall y \in \mathrm{Fix}T$, $\langle y - Tx \mid x - Tx \rangle \leq 0$.

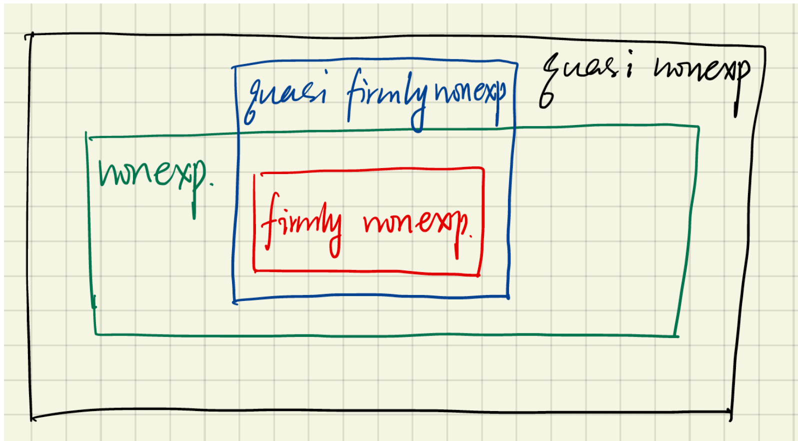

Figure 1: Illustration of relations