Lecture 2

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1 Overview

In this lecture, we introduce the concept of convexity, including convex sets and some convexity-preserving operations.

2 Convex sets

A subset C of $\mathcal{H}$ is **convex** if

- $\forall \alpha \in (0,1), \alpha C + (1-\alpha)C \subset C$.
- That is, $\forall \alpha \in (0,1), \forall x \in C, \forall y \in C,\ \alpha x + (1-\alpha)y \in C$.
- Or equivalently, $\forall x \in C, \forall y \in C,\ (x,y) \subset C$, where (x,y) denotes the segment between x and y.

Examples/counterexamples:

- A ball $B(x;\rho)$ is convex.
- Half space and affine space are convex.
- The union of two disjoint balls are not convex.

3 Convexity-preserving operations

- Let $(C_i)_{i\in I}$ be a family of convex subsets of $\mathcal{H}$, then $\cap_{i\in I} C_i$ is convex.
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- Let $(C_i)_{i\in I}$ be a finite family of convex subsets of $\mathcal{H}$, and let $(\alpha_i)_{i\in I} \in \mathbb{R}$, then $\sum_{i\in I} \alpha_i C_i$ is convex.
- Let $\mathcal{G}$ be a Euclidean space, and $L : \mathcal{H} \to \mathcal{G}$ be a linear operator. Let $C \in \mathcal{H}$ and $D \in \mathcal{G}$ be convex subsets. Then,

- $L(C) = \{Lx \mid x \in C\}$ is convex
- $L^{-1}(D) = \{x \in \mathcal{H} \mid Lx \in D\}$ is convex

Proof: Take x_1 and x_2 in $\mathcal{H}$, $\alpha \in (0,1)$. Set $y_1 = Lx_1$, $y_2 = Lx_2$.

i) Suppose that x_1 and x_2 are in C. Then $y_1 \in L(C)$ and $y_2 \in L(C)$, and

$$\alpha y_1 + (1-\alpha)y_2 = \alpha Lx_1 + (1-\alpha)Lx_2 = L(\alpha x_1 + (1-\alpha)x_2) \in L(C)$$

by the convexity of C. Therefore, $L(C)$ is convex.

ii) Suppose that y_1 and y_2 are in D. Then $Lx_1 \in D$ and $Lx_2 \in D$. By the convexity of D, we have

$$D \ni \alpha Lx_1 + (1-\alpha)Lx_2 = L(\alpha x_1 + (1-\alpha)x_2).$$

Therefore, $L(\alpha x_1 + (1-\alpha)x_2) \in D$ and thus $L^{-1}(D)$ is convex.