Lecture 1

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1 Optimization

Optimization is the science of making better decisions in unclear environments.

— Patrick L. Combettes

2 Overview

In this lecture, we introduce the Euclidean space, some basic identities and inequalities, and some basic concepts based on Euclidean space.

3 Euclidean space

The **Euclidean space** $\mathcal{H}$ is a real finite dimensional vector space equipped with a scalar product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, $(x, y) \mapsto \langle x \mid y \rangle$.

- **Properties of the scalar product**
  - $\forall (x, y) \in \mathcal{H}^2, \langle x \mid y \rangle = \langle y \mid x \rangle$
  - $\forall \alpha \in \mathbb{R}, \forall (x, y, z) \in \mathcal{H}^3, \langle \alpha x + y \mid z \rangle = \alpha \langle x \mid z \rangle + \langle y \mid z \rangle$
  - $\forall x \in \mathcal{H} \setminus \{0\}, \langle x \mid x \rangle > 0$

A **norm** on $\mathcal{H}$ is a function $\|\cdot\|: \mathcal{H} \to \mathbb{R}$, $x \mapsto \|x\|$ such that

(i) $\forall (x, y) \in \mathcal{H}^2, \|x + y\| \leq \|x\| + \|y\|$

(ii) $\forall \alpha \in \mathbb{R}, \forall x \in \mathcal{H}, \|\alpha x\| = \alpha \|x\|$

(iii) $\forall x \in \mathcal{H}, x = 0 \iff \|x\| = 0_{\mathcal{H}}$

The **default norm** of $\mathcal{H}$ is the one associated with the scalar product:

$$\forall x \in \mathcal{H}, \|x\| = \sqrt{\langle x \mid x \rangle}, \tag{1}$$

which also gives rise to a **distance**

$$\forall (x, y) \in \mathcal{H}^2, d(x, y) = \|x - y\| = \sqrt{\langle x - y \mid x - y \rangle}. \tag{2}$$

4 Basic identities and inequalities

- **Cauchy–Schwarz**: $\forall (x, y) \in \mathcal{H}^2, |\langle x \mid y \rangle| \leq \|x\|\|y\|$. Moreover, $|\langle x \mid y \rangle| = \|x\|\|y\| \iff \exists \alpha \in \mathbb{R}_+, x = \alpha y$
- $\forall (x, y) \in \mathcal{H}^2, \|x + y\|^2 = \|x\|^2 + 2\langle x \mid y \rangle + \|x\|^2$.
- **Parallelogram identity**: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|\|y\|^2$
- **Polarization identity**: $\langle x \mid y \rangle = \frac{1}{4}\left[\|x + y\|^2 - \|x - y\|^2\right]$
- Let $\alpha \in \mathbb{R}$, then $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$
- $\langle x \mid y \rangle \leq 0 \iff \forall \alpha \in [0, +\infty), \|x\| \leq \|x - \alpha y\| \iff \forall \alpha \in [0, 1], \|x\| \leq \|x - \alpha y\|$

5 Basic concepts

- Let $x \in \mathcal{H}$ and $\rho \in (0, +\infty)$, the **ball** with center x and radius ρ is $B(x; \rho) = \{y \in \mathcal{H} \mid \|x - y\| \leq \rho\}$.
- Let $C \in \mathcal{H}$, then C is an **open set** if $\forall x \in \mathcal{H}, \exists \rho \in (0, +\infty), B(x; \rho) \in C$. C is a **closed set** if $\mathcal{H} \setminus C$ is open.
- Take a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ and a point $x \in \mathcal{H}$, then x_n **converges to** x if $\|x_n - x\| \to 0$ as $n \to \infty$. In other words, $\forall \rho \in (0, \infty), \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N \Rightarrow x_n \in B(x; \rho)$.
- C **is a closed set** means that for every sequence $(x_n)_{n\in\mathbb{N}}$ in C that converges, say $x_n \to x$, then $x \in C$.

  Ex. Let $\mathcal{H} = \mathbb{R}, C = (0, 1), x_n = \frac{1}{1+n}$, then $x_n \to 0$ which is not in C, so C is not closed.

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ converges if and only if it is a **Cauchy sequence**: $\|x_n - x_m\| \to 0$, as $n, m \to \infty$.
- **Operator**
  - Let $\mathcal{G}$ be an Euclidean space. The notation $T : \mathcal{H} \to \mathcal{G}$ means that the operator (also called mapping) T maps every point x in $\mathcal{H}$ to a point Tx in $\mathcal{G}$.

  - T is a **linear operator** if $\forall \alpha \in \mathbb{R}, \forall (x, y) \in \mathcal{H}^2$, $T(\alpha x + y) = \alpha Tx + Ty$.
  - T is **continuous at** $x \in \mathcal{H}$ if there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$, $x_n \to x \Rightarrow Tx_n \to Tx$. T is **continuous** if this holds for every $x \in \mathcal{H}$.

- Let $\mathbb{R}$ be the real line $(-\infty, +\infty)$ and $I \subset \mathbb{R}$
  - The **inifimum** of I is the greatest lower bound of I ($+\infty$ if $I = \emptyset$).
    * A lower bound of I is a value $a \in \mathbb{R}$ such that $\forall x \in I, a \leq x$ and $\forall \epsilon > 0, \exists x \in I$ such that $x < a + \epsilon$.
  - The **supremum** of I is the lowest upper bound of I ($-\infty$ if $I = \emptyset$).
    * A upper bound of I is a value $b \in \mathbb{R}$ such that $\forall x \in I, x \leq b$ and $\forall \epsilon > 0, \exists x \in I$ such that $x > b - \epsilon$.

  Eg. $\inf(0, 1] = 0$, $\sup(0, 1] = 1$.

  - If $\inf I \in I$, it is called a **minimum**, and it is denoted by $\min I$.
  - If $\sup I \in I$, it is called a **maximum**, and it is denoted by $\max I$.

  Eg. Let $I = \{y \mid y = e^{-x}\}$, then $\inf I = 0$, $\arg\min_x e^{-x}$ does not exist since $0 \notin I$.

- Let C be a subset of $\mathcal{H}$, and let $f : \mathcal{H} \to [-\infty, +\infty]$, $x \mapsto f(x)$, then $f(C) = \{f(x) \mid x \in C\} \subset [-\infty, +\infty]$.
  - Notation: $\inf f(C) = \inf_{x \in C} f(x)$, $\sup f(C) = \sup_{x \in C} f(x)$

- **Rules in** $[-\infty, +\infty]$: $x + y$ is defined in a natural way, but $(+\infty) - (+\infty)$ is not defined. $\forall x \in (-\infty, +\infty], x + (+\infty) = +\infty$.

- Let $C \subset \mathcal{H}$,
  - C is a **vector subspace** if
    i) $C \neq \emptyset$;
    ii) $C + C = \{x + y \mid x \in C, y \in C\} \subset C$;
    iii) $\forall \alpha \in \mathbb{R}, \alpha C = \{\alpha x \mid x \in C\} \subset C$.
  - C is an **affine subspace** if $\forall \alpha \in \mathbb{R}, \alpha C + (1 - \alpha)C = C$.

  Eg. Let $u \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$ and $\eta \in \mathbb{R}$, then $C = \{x \in \mathcal{H} \mid \langle x \mid u \rangle = \eta\}$ is called a **hyperplane**.

  Eg. Let $u \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$ and $\eta \in \mathbb{R}$, then $C = \{x \in \mathcal{H} \mid \langle x \mid u \rangle \leq \eta\}$ is a **closed affine hal-space**, but not an affine space.

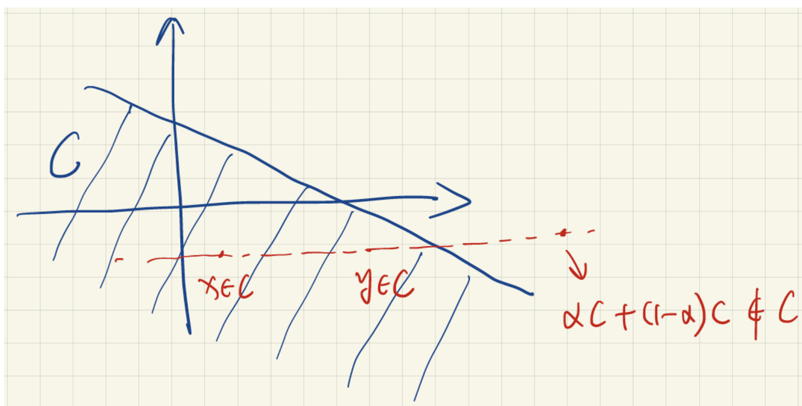

Figure 1: Illustration