A Convex-Nonconvex Strategy for Grouped Variable Selection

Xiaoqian Liu

Department of Statistics

North California State University

Collaborators:
Aaron Molstad, University of Florida
Eric Chi, Rice University

June 9, 2022

Overview

1. Convex-Nonconvex Penalization
   - Motivation
   - Generalized Minimax Concave (GMC) penalty
2. Group GMC for Grouped Variable Selection
   - The group GMC estimator
   - Algorithms for the group GMC model
   - Error bound for the group GMC estimator
   - Simulations and a real data application
3. Discussion

Introduction

Recover a sparse representation:

$$\text{minimize } F(\boldsymbol{\beta}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 + \lambda\psi(\boldsymbol{\beta}) \tag{1}$$

- Statistics – penalized linear regression
  - $\boldsymbol{y} \in \mathbb{R}^n$: response
  - $\boldsymbol{X} \in \mathbb{R}^{n \times p}$: design matrix
  - $\boldsymbol{\beta} \in \mathbb{R}^p$: vector of coefficients
- Signal processing – signal recovery/denoising
  - $\boldsymbol{y} \in \mathbb{R}^n$: vector of observations
  - $\boldsymbol{X} \in \mathbb{R}^{n \times p}$: linear operator
  - $\boldsymbol{\beta} \in \mathbb{R}^p$: signal vector
- $\psi : \mathbb{R}^p \mapsto \mathbb{R}$ – penalty function promoting sparsity in $\boldsymbol{\beta}$.

Convex penalization

Commonly used convex penalties:

- $\psi(\boldsymbol{\beta}) = \|\boldsymbol{\beta}\|_1$
  - Lasso (Tibshirani, 1996)
  - Basis Pursuit (Chen and Donoho, 1994)
- $\psi(\boldsymbol{\beta}) = \alpha\|\boldsymbol{\beta}\|_1 + (1-\alpha)\|\boldsymbol{\beta}\|_2^2$
  - Elastic Net (Zou and Hastie, 2005)

Characteristics of convex penalties:

+ no suboptimal local minimizers
− underestimate large magnitude components

Nonconvex penalization

Commonly used nonconvex penalties:

- the smoothly clipped absolute deviations (SCAD) penalty
  - (Fan and Li, 2001)
- the minimax concave penalty (MCP)
  - (Zhang et al., 2010)

Characteristics of nonconvex penalties:

+ more accurate estimation
− existence of suboptimal local minimizers

Introduction

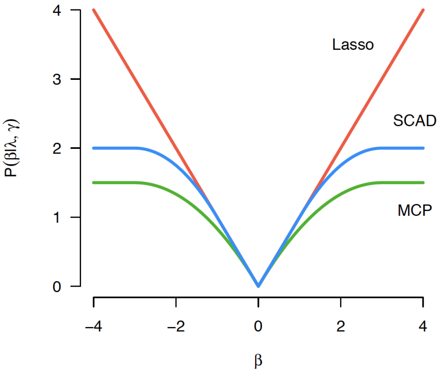


Figure: Visualization of Lasso, SCAD and MCP (Adopted from Patrick Breheny's lecture on BIOS 7240).

- non-differentiability at the origin $\rightarrow$ sparsity

Introduction

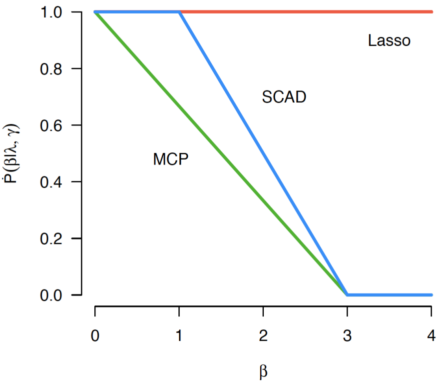


Figure: Visualization of derivatives of Lasso, SCAD and MCP (Adopted from Patrick Breheny's lecture on BIOS 7240)

- derivative → penalization rate (estimation bias)

The GMC penalization

A convex-nonconvex strategy:

Design a nonconvex penalty but maintain the convexity of the problem.

The **GMC penalty** (Selesnick, 2017):

$$\psi_{\boldsymbol{B}}(\boldsymbol{\beta}) \;=\; \|\boldsymbol{\beta}\|_1 - \min_{\boldsymbol{v}\in\mathbb{R}^p}\{\|\boldsymbol{v}\|_1 + \frac{1}{2}\|\boldsymbol{B}(\boldsymbol{\beta}-\boldsymbol{v})\|_2^2\}, \tag{2}$$

where $\boldsymbol{B} \in \mathbb{R}^{n\times p}$ is a matrix parameter.

The GMC penalization

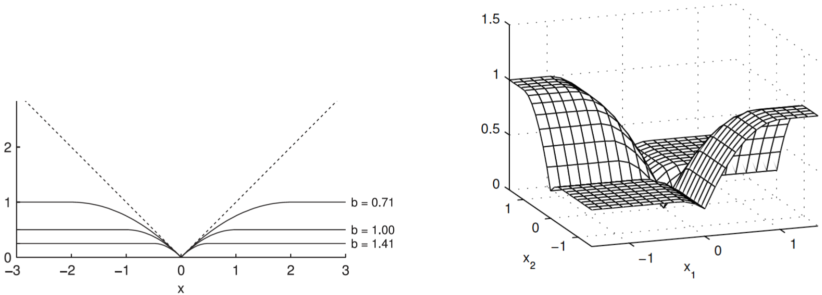


Figure: Visualization of the GMC penalty in the univariate case (left) and the multivariate case (right). Adopted from Selesnick (2017).

The GMC penalization

The optimization problem

$$\text{minimize } F(\boldsymbol{\beta}) = \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 + \lambda\psi_{\boldsymbol{B}}(\boldsymbol{\beta}) \tag{3}$$

maintains convex if

$$\boldsymbol{X}^\top \boldsymbol{X} \succeq \lambda \boldsymbol{B}^\top \boldsymbol{B}. \tag{4}$$

- **Convexity-preserving condition** for the GMC model (3).

The GMC penalization

An open question for the GMC penalization:

how to set the matrix parameter $\boldsymbol{B}$?

An approach in (Selesnick, 2017):

$$\boldsymbol{B} = \sqrt{\theta/\lambda}\boldsymbol{X}, \quad \text{with } \theta \in (0, 1),$$

then $\lambda \boldsymbol{B}^\top \boldsymbol{B} = \theta \boldsymbol{X}^\top \boldsymbol{X}$, which satisfies condition (4).

Grouped variable selection

The classical linear regression setting:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

- $\boldsymbol{y} \in \mathbb{R}^n$: response vector
- $\boldsymbol{X} \in \mathbb{R}^{n \times p}$: design matrix whose columns are p covariate variables
  with natural group structures
  e.g. categorical data analysis
- $\boldsymbol{\epsilon} \in \mathbb{R}^n$: vector of noise variables with mean zero and variance σ^2

grouped variable selection and coefficient estimation

- Convex penalization
  Group Lasso (Yuan and Lin, 2006) and its variants

$$\hat{\boldsymbol{\beta}}_{\text{grLasso}} = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2n}\|\boldsymbol{y} - \sum_{j=1}^{J} \boldsymbol{X}_j\boldsymbol{\beta}_j\|_2^2 + \lambda \sum_{j=1}^{J} K_j\|\boldsymbol{\beta}_j\|_2 \tag{5}$$

  - $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^T, ..., \boldsymbol{\beta}_J^T)^T \in \mathbb{R}^p$ with $\boldsymbol{\beta}_j \in \mathbb{R}^{p_j}$ and $\sum_{j=1}^J p_j = p$
  - $\boldsymbol{X}_j$: submatrix of $\boldsymbol{X}$ $\rightarrow$ variables in the j-th group
  - K_js: adjusting for the group sizes, e.g. $K_j = \sqrt{p_j}$
- Nonconvex penalization
  Group SCAD (Wang et al., 2007), Group MCP (Huang et al., 2012)

The group GMC estimator

The **group GMC penalty** (Liu et al., 2021):

$$\phi_{\boldsymbol{B}}(\boldsymbol{\beta}) = \sum_{j=1}^{J} K_j \|\boldsymbol{\beta}_j\|_2 - \min_{\boldsymbol{v} \in \mathbb{R}^p} \left\{ \sum_{j=1}^{J} K_j \|\boldsymbol{v}_j\|_2 + \frac{1}{2n} \|\boldsymbol{B}(\boldsymbol{\beta} - \boldsymbol{v})\|_2^2 \right\} \qquad (6)$$

- $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^T, ..., \boldsymbol{\beta}_J^T)^T \in \mathbb{R}^p$
- $\boldsymbol{v} = (\boldsymbol{v}_1^T, ..., \boldsymbol{v}_J^T)^T \in \mathbb{R}^p$
- For each j, $\boldsymbol{\beta}_j, \boldsymbol{v}_j \in \mathbb{R}^{p_j}$ with $\sum_{j=1}^{J} p_j = p$

The group GMC estimator

The group GMC model:

$$\arg\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{1}{2n}\|\boldsymbol{y} - \boldsymbol{X}\beta\|_2^2 + \lambda\phi_{\boldsymbol{B}}(\beta), \tag{7}$$

- $\lambda \geq 0$: tuning parameter representing the degree of penalization
- $\boldsymbol{B}$: matrix parameter controlling the concavity of the penalty

The group GMC estimator

The group GMC problem (7) is convex if

$$\boldsymbol{X}^\top \boldsymbol{X} \succeq \lambda \boldsymbol{B}^\top \boldsymbol{B} \tag{8}$$

- **convexity-preserving condition** for group GMC

The group GMC estimator

Set matrix $\boldsymbol{B}$ for group GMC:

$$\lambda \boldsymbol{B}^{\mathsf{T}} \boldsymbol{B} = \theta \boldsymbol{X}^{\mathsf{T}} \boldsymbol{X}, \quad \theta \in [0, 1]. \tag{9}$$

- θ: the **convexity-preserving parameter** of the group GMC model
  - $\theta = 0$: group GMC $\rightarrow$ group Lasso
  - $\theta = 1$: a maximally nonconvex penalty

The group GMC estimator

Relation between group GMC and group MCP (Huang et al., 2012):

Remark

The group GMC method is equivalent to the group MCP method when $\mathbf{B}^{\mathsf{T}}\mathbf{B}$ *is diagonal and the diagonal elements are suitably designed. This equivalence also holds for the GMC and MCP.*

Properties of the solution path:

Theorem

Suppose $\boldsymbol{X}^\top \boldsymbol{X} \succ \lambda \boldsymbol{B}^\top \boldsymbol{B}$, *then the solution path* $\beta^\star(\lambda)$ *to the group GMC problem* (7) *exists, is unique, and is continuous in* λ.

- Problem (7) is well-posed
- Warm start when solving a sequence of problems over a grid of λ values

Properties of the solution path:

Theorem

The group GMC problem (7) *has a unique solution* $\beta^\star(\lambda) = \mathbf{0}$ *for all* λ *greater than* $\lambda_0 = \max_j \left\{ \frac{\|\boldsymbol{X}_j^\top \boldsymbol{y}\|_2}{nK_j} \right\}$, *where* $\boldsymbol{X}_j$ *and* K_j *are as defined in* (5) *for* $j = 1, \cdots, J$.

- A precise range of λ, $[0, \lambda_0]$, to sample the full dynamic range of the coefficient estimation

Algorithms for the group GMC model

Recast problem (7) as a saddle-point problem

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^p} \max_{\boldsymbol{v}\in\mathbb{R}^p} f(\beta) + \beta^\mathsf{T} \boldsymbol{Z}\boldsymbol{v} - g(\boldsymbol{v}), \tag{10}$$

where

$$f(\beta) = \frac{1}{2n}\|\boldsymbol{y} - \boldsymbol{X}\beta\|_2^2 + \lambda \sum_{j=1}^{J} K_j \|\boldsymbol{\beta}_j\|_2 - \frac{\lambda}{2n}\|\boldsymbol{B}\beta\|_2^2,$$

$$g(\boldsymbol{v}) = \frac{\lambda}{2n}\|\boldsymbol{B}\boldsymbol{v}\|_2^2 + \lambda \sum_{j=1}^{J} K_j \|\boldsymbol{v}_j\|_2,$$

$$\boldsymbol{Z} = \frac{\lambda}{n}\boldsymbol{B}^\mathsf{T}\boldsymbol{B}.$$

- Primal-Dual Hybrid Gradient (PDHG) method (Goldstein et al., 2013, 2015a)

Algorithms for the group GMC model

Algorithm 1 Basic PDHG steps for problem (10)

Set $\beta_0 \in \mathbb{R}^p, \boldsymbol{v}_0 \in \mathbb{R}^p, \sigma_k > 0, \tau_k > 0$
for $k = 1$ to K **do**
  $\hat{\boldsymbol{\beta}}_{k+1} = \boldsymbol{\beta}_k - \tau_k \boldsymbol{Z}^T \boldsymbol{v}_k$
  $\boldsymbol{\beta}_{k+1} = \arg\min_{\boldsymbol{\beta} \in \mathbb{R}^p} f(\boldsymbol{\beta}) + \frac{1}{2\tau_k} \|\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_{k+1}\|_2^2$
  $\hat{\boldsymbol{v}}_{k+1} = \boldsymbol{v}_k + \sigma_k \boldsymbol{Z}(2\boldsymbol{\beta}_{k+1} - \boldsymbol{\beta}_k)$
  $\boldsymbol{v}_{k+1} = \arg\min_{\boldsymbol{v} \in \mathbb{R}^p} g(\boldsymbol{v}) + \frac{1}{2\sigma_k} \|\boldsymbol{v} - \hat{\boldsymbol{v}}_{k+1}\|_2^2$
end for

Algorithms for the group GMC model

Updating β_{k+1} and $\boldsymbol{v}_{k+1}$:

$$\beta_{k+1} = \underset{\beta \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \frac{1}{2n} \|\boldsymbol{y} - \boldsymbol{X}\beta\|_2^2 - \frac{\lambda}{2n} \|\boldsymbol{B}\beta\|_2^2 + \frac{1}{2\tau_k} \|\beta - \hat{\beta}_{k+1}\|_2^2 \right\} + \lambda \sum_{j=1}^{J} K_j \|\beta_j\|_2$$

$$\boldsymbol{v}_{k+1} = \underset{\boldsymbol{v} \in \mathbb{R}^p}{\operatorname{argmin}} \left\{ \frac{\lambda}{2n} \|\boldsymbol{B}\boldsymbol{v}\|_2^2 + \frac{1}{2\sigma_k} \|\boldsymbol{v} - \hat{\boldsymbol{v}}_{k+1}\|_2^2 \right\} + \lambda \sum_{j=1}^{J} K_j \|\boldsymbol{v}_j\|_2$$

- group Lasso penalized problems
- Fast Adaptive Shrinkage/Thresholding Algorithm (FASTA) (Goldstein et al., 2014, 2015b)

Some definitions:

- $\mathbf{v}^\star = \operatorname*{argmin}_{\mathbf{v}\in\mathbb{R}^p} \left\{ \sum_{j=1}^{J} K_j \|\mathbf{v}_j\|_2 + \frac{1}{2n}\|\mathbf{B}(\beta^\star - \mathbf{v})\|_2^2 \right\}$
- $\mathcal{S} := \{j : \|\beta_j^\star\|_2 \neq 0, j \in [J]\}$ and $\mathcal{S}^c := [J] \setminus \mathcal{S}$
- $$\nu_j = \begin{cases} K_j + n^{-1}\|[\mathbf{B}^\top\mathbf{B}]_{j,\cdot}(\beta^\star - \mathbf{v}^\star)\|_2, & j \in \mathcal{S} \\ K_j - n^{-1}\|[\mathbf{B}^\top\mathbf{B}]_{j,\cdot}(\beta^\star - \mathbf{v}^\star)\|_2, & j \in \mathcal{S}^c \end{cases}$$
- $\bar{\nu} := \max_{j\in\mathcal{S}} \nu_j$ and $\underline{\nu} := \min_{k\in\mathcal{S}^c} \nu_k$

Error bound for the group GMC estimator

Conditions and assumptions:

- $\mathbf{X}$ satisfies a "block-normalization" condition:

$$\|\mathbf{X}_{\cdot,j}\| \leq \sqrt{n}, \quad j \in [J]$$

- **A1.** (Subgaussian errors). The data are generated from (12) where $\boldsymbol{\epsilon} \in \mathbb{R}^n$ has independent entries which are σ-subgaussian random variables for $0 < \sigma < \infty$. That is, $\mathbb{E}(\epsilon_i) = 0$ and for all $t \in \mathbb{R}$, $\mathbb{E}\{\exp(t\epsilon_i)\} \leq \exp(t^2\sigma^2/2)$ for each $i \in [n]$.
- **A2.** (Convexity) The matrix $\mathbf{B}$ is chosen so that $\mathbf{X}^\mathsf{T}\mathbf{X} \succeq \lambda \mathbf{B}^\mathsf{T}\mathbf{B}$.
- **A3.** (Sample size) The sample size n is sufficiently large so that $\nu_k > 0$ for all $k \in \mathcal{S}^c$.

Conditions and assumptions:

- **A4.** (Restricted eigenvalue condition) For a fixed $c > 1$, define

$$\mathbb{C}_n(\mathcal{S}, \nu, c) = \left\{ \boldsymbol{\Delta} \in \mathbb{R}^p : \boldsymbol{\Delta} \neq \mathbf{0}, \sum_{k \in \mathcal{S}^c} \left(\nu_k - \frac{\nu}{c} \right) \|\boldsymbol{\Delta}_k\|_2 \leq \sum_{j \in \mathcal{S}} \left(\nu_j + \frac{\nu}{c} \right) \|\boldsymbol{\Delta}_j\|_2 \right\}.$$

We assume there exists a constant $k > 0$ such that for all n and p,

$$0 < k \leq \kappa_{\mathbf{B}}(\mathcal{S}, c) = \inf_{\boldsymbol{\Delta} \in \mathbb{C}_n(\mathcal{S}, \nu, c)} \frac{\boldsymbol{\Delta}^{\mathsf{T}}(\mathbf{X}^{\mathsf{T}}\mathbf{X} - \lambda \mathbf{B}^{\mathsf{T}}\mathbf{B})\boldsymbol{\Delta}}{2n\|\boldsymbol{\Delta}\|_2^2}.$$

Error bound for the group GMC estimator

Theorem

(Error bound for group GMC) Let $c > 1$ *and* $k_1 > 0$ *be fixed constants. If assumptions* **A1–A4** *hold and*

$$\lambda = \frac{2c\sigma}{\underline{\nu}} \left(\max_{j \in [J]} \sqrt{\frac{p_j}{n}} + \sqrt{\frac{k_1 \log(J)}{n}} \right),$$

then with probability at least $1 - 2\exp(-2k_1 \log(J))$*,*

$$\|\hat{\beta}(\lambda) - \beta^\star\|_2 \leq \frac{2c\sigma}{\kappa_{\mathbf{B}}(\mathcal{S}, c)} \left(\frac{\bar{\nu}}{\underline{\nu}} + \frac{1}{c} \right) \left\{ \left(\max_{j \in [J]} \sqrt{\frac{|\mathcal{S}| p_j}{n}} \right) + \sqrt{\frac{|\mathcal{S}| k_1 \log(J)}{n}} \right\},$$

where $\hat{\beta}(\lambda)$ *is the group GMC estimator obtained from* (7).

Error bound for the group GMC estimator

- Same asymptotic error rate as the group Lasso estimator
- Choose $\mathbf{B}$ such that $\kappa_{\mathbf{B}}(\mathcal{S}, c)$ is large and $\bar{\nu}/\underline{\nu}$ is small

Error bound for the group GMC estimator

Theorem

(Error bound for GMC) Let $c > 1$ and $k_2 \in (0, 1/2)$ be fixed constants. Let $p_j = 1$ for $j \in [p]$ so that $\mathcal{S} = \{j : \beta_j^\star \neq 0, j \in [p]\}$. If assumptions **A1–A4** *hold and $\lambda = (c\sigma/\underline{\nu})\sqrt{2\log(p/k_2)/n}$, then with probability at least $1 - 2k_2$,*

$$\|\hat{\beta}(\lambda) - \beta^\star\|_2 \leq \frac{c\sigma}{\kappa_{\mathbf{B}}(\mathcal{S}, c)} \left(\frac{\bar{\nu}}{\underline{\nu}} + \frac{1}{c}\right) \sqrt{\frac{2|\mathcal{S}|\log(p/k_2)}{n}},$$

where $\hat{\beta}(\lambda)$ is the corresponding GMC estimator.

Simulation Experiments

- Models:
  - an ANOVA model with all two-way interactions
  - an additive model including both categorical and continuous variables
- Factors of interest:
  - signal-to-noise ratio (SNR) of the model
  - correlation among groups
  - problem dimension
  - convexity-preserving parameter (for the group GMC)

Simulation experiments

Data generation of the ANOVA model:

- Z_1, Z_2, Z_3 and Z_4 from a centered multivariate normal distribution
  - $Cov(Z_i, Z_j) = \rho^{|i-j|}$
- $Z_1, \cdots, Z_4$ are trichotomized to $0, 1$ or 2
  - 0 if smaller than $\Phi^{-1}(\frac{1}{3})$
  - 1 if larger than $\Phi^{-1}(\frac{1}{3})$
  - 2 if in between

Simulation experiments

Data generation of the ANOVA model:

$$
\begin{aligned}
y = 3\mathbb{1}(Z_1 = 1) + 2\mathbb{1}(Z_1 = 0) + 3\mathbb{1}(Z_2 = 1) + 2\mathbb{1}(Z_2 = 0) + \\
\mathbb{1}(Z_1 = 1, Z_2 = 1) + \mathbb{1}(Z_1 = 1, Z_2 = 0) + \\
2\mathbb{1}(Z_1 = 0, Z_2 = 1) + 2.5\mathbb{1}(Z_1 = 0, Z_2 = 0) + \epsilon,
\end{aligned}
$$

- $\mathbb{1}(\cdot)$ is the indicator function
- $\epsilon \sim N(0, \sigma^2)$
- 32 covariate variables from 10 groups

Simulation experiments

Performance in three aspects:

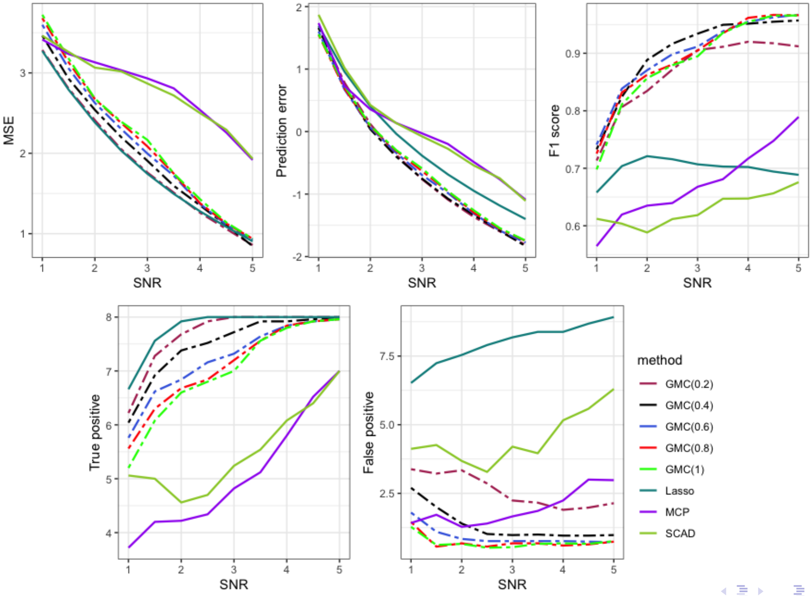
- Coefficient estimation
  - SE $= \|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\|_2^2$
- Prediction performance
  - prediction error $= \frac{1}{n}\|\boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2$
- Support recovery
  - F1 score $= \dfrac{2TP}{2TP + FP + FN}$
  - true positive (TP) and false positive (FP)

		Estimation	
		$\hat{\beta}_j != 0$	$\hat{\beta}_j = 0$
Truth	$\beta_j != 0$	TP	FN
	$\beta_j = 0$	FP	TN

Simulation experiments

- **Case I: effect of the SNR**
  - uncorrelated groups ($\rho = 0$)
  - problem dimension $p = 32$
  - sample size $n = 100$
  - SNR $\in \{1, 2, \cdots, 5\}$
  - $\theta \in \{0.2, 0.4, 0.6, 0.8, 1\}$

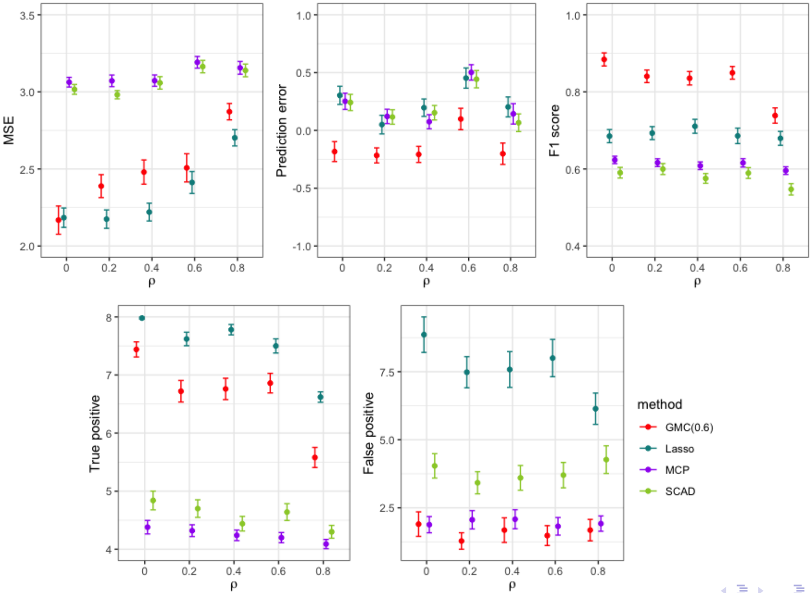
Simulation experiments



Simulation experiments

- **Case II: effect of the correlation among groups**
  - SNR $= 2$
  - problem dimension $p = 32$
  - sample size $n = 100$
  - $\theta = 0.6$
  - correlation $\rho \in \{0, 0.2, 0.4, 0.6, 0.8\}$

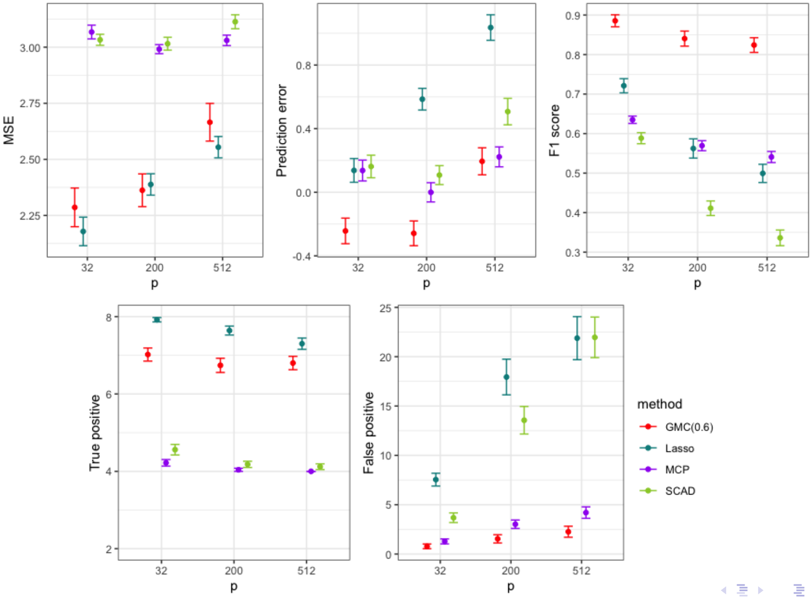
Simulation experiments



Simulation experiments

- **Case III: effect of the problem dimension**
  - uncorrelated groups ($\rho = 0$)
  - SNR $= 2$
  - sample size $n = 100$
  - $\theta = 0.6$
  - $p \in \{32, 200, 512\}$

Simulation experiments



Real data application

The birth weight data set investigated in Yuan and Lin (2006):

- risk factors associated with low rank infant birth weight
- 189 observations of a response variable (infant birth weight)
- 8 explanatory variables (continuous and categorical)

Real data application

Table 1. *Description of the birth weight data set*

Name	Type	Variable description
Birth weight	Continuous	Infant birth weight in kilograms
Mother's age	Continuous	Mother's age in years
Mother's weight	Continuous	Mother's weight in pounds at last menstrual period
Race	Categorical	Mother's race (white, black or other)
Smoking	Categorical	Smoking status during pregnancy (yes or no)
# Premature	Categorical	Previous premature labors (0, 1, or more)
Hypertension	Categorical	History of hypertension (yes or no)
Uterine irritability	Categorical	Presence of uterine irritability (yes or no)
# Phys. visits	Categorical	Number of physician visits during the first trimester (0, 1, 2, or more)

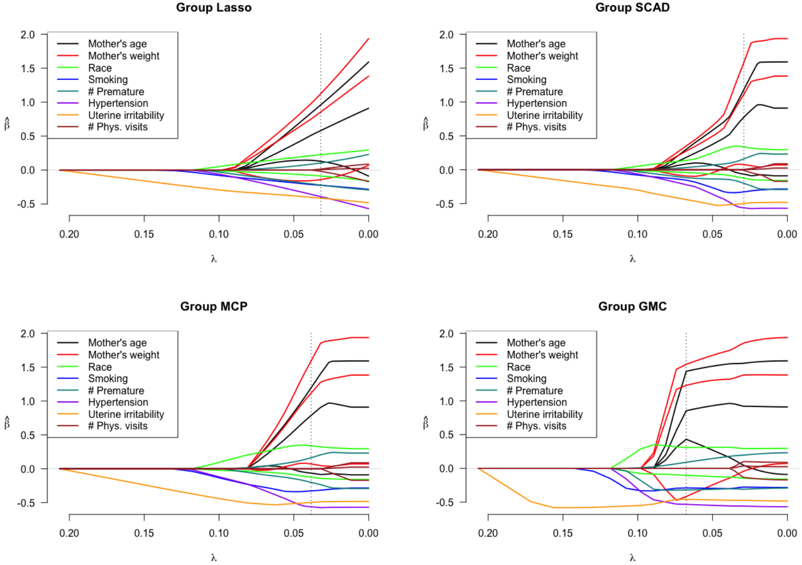
- 16 covariate variables from 8 groups

Real data application

Table 2. *Summarized results for the birth weight data*

	Prediction error	# nonzero groups	Excluded groups
Group Lasso	0.36	8	none
Group SCAD	0.35	8	none
Group MCP	0.35	7	# Phys. visits
Group GMC	0.35	7	# Phys. visits

Real data application



Discussion

Summary:

- A group GMC method for grouped variable selection and coefficient estimation in linear regression
- Convexity preserving condition, relation to existing methods, and properties of solution path
- Algorithms for computing the solution path
- Error bounds of the (group) GMC estimator
- Simulations and a real data application

Discussion

Future directions:

- Guidance on setting the matrix parameter $\boldsymbol{B}$
- Extension to generalized linear models
- Computation of the (group) GMC problem

Contact

Please reach out if you have any questions:

- Email: xliu62@ncsu.edu
- Website: https://xiaoqian-liu.github.io/

Useful links:

- The original GMC paper:
  https://ieeexplore.ieee.org/document/7938377
- Matlab code for GMC:
  https://codeocean.com/capsule/2729219/tree/v1
- Our group GMC paper:
  https://arxiv.org/abs/2111.15075
- An R package for group GMC will be available on CRAN soon

Thank You!

Reference I

Chen, S. and Donoho, D. (1994). Basis pursuit. In *Proceedings of 1994 28th Asilomar Conference on Signals, Systems and Computers*, volume 1, pages 41–44. IEEE.

Fan, J. and Li, R. (2001). Variable selection via nonconcave penalized likelihood and its oracle properties. *Journal of the American statistical Association*, 96(456):1348–1360.

Goldstein, T., Li, M., and Yuan, X. (2015a). Adaptive primal-dual splitting methods for statistical learning and image processing. *Advances in Neural Information Processing Systems*, 28:2089–2097.

Goldstein, T., Li, M., Yuan, X., Esser, E., and Baraniuk, R. (2013). Adaptive primal-dual hybrid gradient methods for saddle-point problems. *arXiv preprint arXiv:1305.0546*.

Goldstein, T., Studer, C., and Baraniuk, R. (2014). A field guide to forward-backward splitting with a FASTA implementation. *arXiv eprint*, abs/1411.3406.

Reference II

Goldstein, T., Studer, C., and Baraniuk, R. (2015b). FASTA: A generalized implementation of forward-backward splitting. http://arxiv.org/abs/1501.04979.

Huang, J., Breheny, P., and Ma, S. (2012). A selective review of group selection in high-dimensional models. *Statistical science: a review journal of the Institute of Mathematical Statistics*, 27(4).

Liu, X., Molstad, A. J., and Chi, E. C. (2021). A convex-nonconvex strategy for grouped variable selection. *arXiv preprint arXiv:2111.15075*.

Selesnick, I. (2017). Sparse regularization via convex analysis. *IEEE Transactions on Signal Processing*, 65(17):4481–4494.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(1):267–288.

Reference III

Wang, L., Chen, G., and Li, H. (2007). Group scad regression analysis for microarray time course gene expression data. *Bioinformatics*, 23(12):1486–1494.

Yuan, M. and Lin, Y. (2006). Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(1):49–67.

Zhang, C.-H. et al. (2010). Nearly unbiased variable selection under minimax concave penalty. *The Annals of statistics*, 38(2):894–942.

Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net. *Journal of the royal statistical society: series B (statistical methodology)*, 67(2):301–320.